Abrashkin's work on the higher ramification filtration
Part 1: Background

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Notation

Let K be a local field, complete w.r.t. the valuation $v_K : K^\times \twoheadrightarrow \mathbb{Z}$. Define

$$\mathcal{O}_K = \{x \in K : v_K(x) \geq 0\}$$
$$\mathcal{P}_K = \{x \in K : v_K(x) > 0\}.$$

Assume that $k = \mathcal{O}_K/\mathcal{P}_K$ is a finite field of characteristic p.

If $\mathrm{char}(K) = 0$ then K is a finite extension of $\mathbb{Q}_p$.
If $\mathrm{char}(K) = p$ then $K \cong k((t))$.

Let π_K be a uniformizer for K. Then $v_K(\pi_K) = 1$ and $\mathcal{P}_K = \pi_K\mathcal{O}_K$.

Let L/K be a finite separable extension. Then L is a local field, and v_K extends to a valuation on L, which we also denote by v_K.

Let $e_{L/K} = v_L(\pi_K)$ denote the ramification index of L/K. Then

$$v_K(L^\times) = \frac{1}{e_{L/K}} \cdot \mathbb{Z}.$$

Say the extension L/K is unramified if $e_{L/K} = 1$; say L/K is totally ramified if $e_{L/K} = [L : K]$.

Local class field theory

Let K^{sep}/K be a separable closure of K and let K^{ab}/K be the largest subextension of K^{sep}/K which is Galois with abelian Galois group.

Then there is a continuous one-to-one homomorphism

$$\omega_K : K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$$

with dense image, known as the reciprocity map.

Let L/K be a finite subextension of K^{ab}/K. Then ω_K induces an isomorphism

$$\omega_{L/K} : K^\times/\mathrm{N}_{L/K}(L^\times) \longrightarrow \mathrm{Gal}(L/K).$$

Furthermore, $L/K \mapsto \mathrm{N}_{L/K}(L^\times)$ gives a one-to-one correspondence between finite subextensions L/K of K^{ab}/K and closed subgroups of $K^\times$ with finite index.

Filtrations and class field theory

$K^\times$ has a natural filtration by closed subgroups

$$K^\times \supset \mathcal{O}_K^\times \supset U_K^{(1)} \supset U_K^{(2)} \supset \cdots ,$$

where $U_K^{(n)} = 1 + \mathcal{P}_K^n$. For notational convenience we set $U_K^{(0)} = \mathcal{O}_K^\times$.

Let L/K be a finite subextension of K^{ab}/K and set $G = \mathrm{Gal}(L/K)$. For $n \geq 0$ let $G^{(n)} = \omega_{L/K}(U_K^{(n)})$. Then we get a filtration

$$G \supset G^{(0)} \supset G^{(1)} \supset G^{(2)} \supset \cdots$$

of G which coincides with the "upper ramification filtration".

What this means is the following: Let $\sigma \in G^{(0)}$ with $\sigma \neq \mathrm{id}_L$, and let n be maximum such that $\sigma \in G^{(n)}$. If n is small then $\sigma(\pi_L)$ is far from π_L, while if n is large then $\sigma(\pi_L)$ is close to π_L.

Witt's theorem

Let K be a local field of characteristic p.

Theorem (Witt [Wi36])

Let G be a finite p-group and let $N \trianglelefteq G$. Set $\overline{G} = G/N$ and let L/K be a $\overline{G}$-extension. Then there exists an extension M/L such that M/K is Galois and there is an isomorphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Gal(M/L) & \longrightarrow & Gal(M/K) & \longrightarrow & Gal(L/K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & 1. \end{array}$$

Let $K[p]/K$ be the compositum of all the finite Galois subextensions of $\mathrm{Gal}(K^{sep}/K)$ whose Galois groups are p-groups. Then $K[p]/K$ is the largest Galois subextension of K^{sep}/K whose Galois group is a pro-p group.

Corollary

$Gal(K[p]/K)$ is a free pro-p group.

A subextension of $K[p]/K$

Let $K(p)/K$ be the largest Galois subextension of $K[p]/K$ such that

- $\mathrm{Gal}(K(p)/K)$ has nilpotence class $< p$.
- $\mathrm{Gal}(K(p)/K)$ has exponent p.

Then $\mathrm{Gal}(K(p)/K)$ is free in the category of pro-p groups with nilpotence class $< p$ and exponent dividing p.

In [Ab95] and [Ab21] Abrashkin gave explicit descriptions of the ramification filtration of $\mathrm{Gal}(K(p)/K)$.

Higher ramification theory

Let K be a local field and let L/K be a finite Galois subextension of K^{sep}/K. Let $G = \mathrm{Gal}(L/K)$, let L_0/K be the maximal unramified subextension of L/K, and set $G_0 = \mathrm{Gal}(L/L_0)$. For nonnegative real v define

$$G_v = \{\sigma \in G_0 : v_L(\sigma(\pi_L) - \pi_L) \geq v + 1\}.$$

Say G_v is the vth lower ramification subgroup of G.

We have the following:

- $G_v \trianglelefteq G$.
- If $0 \leq w \leq v$ then $G_v \leq G_w$.
- $G_v = \{\mathrm{id}_L\}$ for all sufficiently large v.
- Let M/K be a subextension of L/K and set $H = \mathrm{Gal}(L/M)$. Then $H_v = H \cap G_v$.

Say $\ell \geq 0$ is a lower ramification break of L/K if $G_{\ell+\epsilon} \lneqq G_\ell$ for all $\epsilon > 0$. This holds if and only if there is $\sigma \in G$ such that $v_L(\sigma(\pi_L) - \pi_L) = \ell + 1$.

The upper numbering for ramification groups

Define the Hasse-Herbrand function $\phi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\phi_{L/K}(x) = \frac{1}{[L : L_0]} \int_0^x |G_t| \, dt.$$

Then $\phi_{L/K}$ is a bijection, so we can define $\psi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\psi_{L/K} = \phi_{L/K}^{-1}$.

For $y \geq 0$ set $G^y = G_{\psi_{L/K}(y)}$. Say G^y is the yth upper ramification subgroup of G.

Say $u \geq 0$ is an upper ramification break of L/K if $G^{u+\epsilon} \lneqq G^u$ for all $\epsilon > 0$. This holds if and only if $\psi_{L/K}(u)$ is a lower ramification break of L/K.

The upper numbering and quotients of Galois groups

Let M/K be a Galois subextension of L/K and set $H = \mathrm{Gal}(L/M)$; then $\mathrm{Gal}(M/K) \cong G/H$. We have the following:

- $\phi_{L/K} = \phi_{M/K} \circ \phi_{L/M}$
- $\psi_{L/K} = \psi_{L/M} \circ \psi_{M/K}$
- $(G/H)^y = G^y H/H$

Hence the upper numbering on ramification subgroups is compatible with quotient groups and subextensions. In particular, if u is an upper ramification break of M/K then u is also an upper ramification break of L/K.

Let F/K be an infinite Galois subextension of K^{sep}/K. Then there are finite Galois subextensions F_n/K of F/K such that $F_1 \subset F_2 \subset \cdots$ and $F = \bigcup_{n \geq 1} F_n$.

Set $G = \mathrm{Gal}(F/K)$ and $H(n) = \mathrm{Gal}(F/F_n)$ for $n \geq 1$. We can define an upper ramification filtration on G by setting

$$G^y = \varprojlim \, (G/H(n))^y$$

for $y \geq 0$.

An example

Let $K = \mathbb{Q}_2(\zeta_4)$; then $\pi_K = \zeta_4 - 1$ is a uniformizer for K. Let $\pi_L = \pi_K^{1/4}$ and set $L = K(\pi_L)$. Then π_L is a uniformizer for L.

L/K is a cyclic extension of degree 4, with $\mathrm{Gal}(L/K) = \langle \sigma \rangle$ satisfying

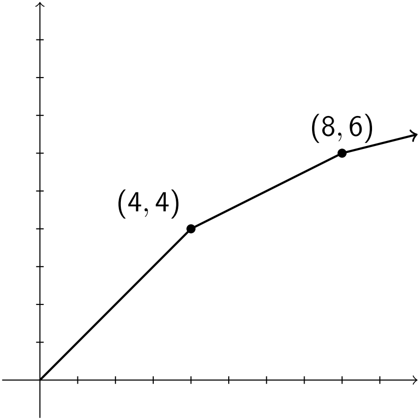
$$\begin{aligned}
\sigma(\pi_L) &= \zeta_4 \pi_L = (1 + \pi_K)\pi_L = \pi_L(1 + \pi_L^4) \\
\sigma^2(\pi_L) &= (1 + \pi_K)^2 \pi_L = \pi_L(1 + \pi_L^8 + 2\pi_L^4) \\
\sigma^3(\pi_L) &= (1 + \pi_K)^3 \pi_L = \pi_L(1 + 3\pi_L^4 + 3\pi_L^8 + \pi_L^{12}).
\end{aligned}$$

Hence $v_L(\sigma(\pi_L) - \pi_L) = v_L(\sigma^3(\pi_L) - \pi_L) = 5$ and $v_L(\sigma^2(\pi_L) - \pi_L) = 9$. Therefore

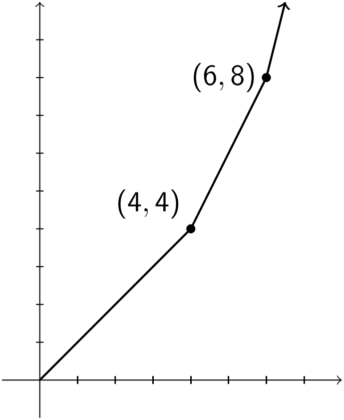
$$G_x = \begin{cases} \langle \sigma \rangle & (0 \le x \le 4) \\ \langle \sigma^2 \rangle & (4 < x \le 8) \\ \{\mathrm{id}_L\} & (8 < x). \end{cases} \qquad G^y = \begin{cases} \langle \sigma \rangle & (0 \le x \le 4) \\ \langle \sigma^2 \rangle & (4 < x \le 6) \\ \{\mathrm{id}_L\} & (6 < x). \end{cases}$$

Thus L/K has lower ramification breaks 4, 8, and upper breaks 4, 6.

Hasse-Herbrand functions for L/K



$y = \phi_{L/K}(x)$



$y = \psi_{L/K}(x)$

Artin-Schreier extensions

Let $K = k((t))$ be a local field of characteristic p, and let $r \in K$ satisfy $v_K(r) = -b < 0$ with $p \nmid b$.

Let $\alpha \in K^{sep}$ be a root of $g(X) = X^p - X - r$ and set $L = K(\alpha)$. Then by Artin-Schreier theory L/K is a cyclic extension of degree p, and there is a generator σ for $\mathrm{Gal}(L/K)$ such that $\sigma(\alpha) = \alpha + 1$.

Since $pv_L(\alpha) = v_L(\alpha^p) = v_L(r)$ the extension L/K is totally ramified. Since $v_L(\alpha) = -b$ and $p \nmid b$ it follows that b is the unique lower ramification break of L/K. Hence b is also the unique upper ramification break of L/K.

If $v_K(r) < 0$ but $p \mid v_K(r)$ then $X^p - X - r$ may or may not be irreducible. Suppose $X^p - X - r$ is irreducible and let L be generated over K by a root of $X^p - X - r$. Then the method described above is not sufficient to determine the ramification break u of L/K, but we do have $u < -v_K(r)$.

Filtered unipotent groups over $\mathbb{F}_p$

Let $\mathcal{G}$ be an algebraic group defined over $\mathbb{F}_p$. Assume further that there are polynomials

$$D_i \in \mathbb{F}_p[X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_{i-1}]$$

such that the group operation is given by

$$\vec{X} * \vec{Y} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} * \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} X_1 + Y_1 \\ X_2 + Y_2 + D_2 \\ \vdots \\ X_n + Y_n + D_n \end{bmatrix}.$$

We say that $\mathcal{G}$ is a *filtered unipotent group* over $\mathbb{F}_p$.

Let $\vec{\beta} \in \mathcal{G}(K) \cong K^n$. We wish to consider the system of equations $\phi(\vec{X}) = \vec{\beta} * \vec{X}$, where ϕ acts as the p-Frobenius on coordinates.

Extensions associated to algebraic groups

Suppose $\vec{x} \in \mathcal{G}(K^{sep}) \cong (K^{sep})^n$ is a solution to $\phi(\vec{X}) = \vec{\beta} * \vec{X}$. Then $x_1^p - x_1 = \beta_1$, and for $2 \leq i \leq n$ we have $x_i^p - x_i = \beta_i + d_i$, where

$$d_i = D_i(\beta_1, \ldots, \beta_{i-1}, x_1, \ldots, x_{i-1}).$$

It follows that the system $\phi(\vec{X}) = \vec{\beta} * \vec{X}$ has p^n distinct solutions in $(K^{sep})^n$. Set $L = K(x_1, \ldots, x_n)$. Then for $\vec{c} \in \mathcal{G}(\mathbb{F}_p)$ we have

$$\phi(\vec{x} * \vec{c}) = \phi(\vec{x}) * \phi(\vec{c}) = (\vec{\beta} * \vec{x}) * \vec{c} = \vec{\beta} * (\vec{x} * \vec{c}).$$

Hence $\vec{x} * \vec{c}$ is also a solution to $\phi(\vec{X}) = \vec{\beta} * \vec{X}$. Since $|\mathcal{G}(\mathbb{F}_p)| = p^n$ the p^n distinct solutions to $\phi(\vec{X}) = \vec{\beta} * \vec{X}$ are given by $\{\vec{x} * \vec{c} : \vec{c} \in \mathcal{G}(\mathbb{F}_p)\}$. Since $\vec{x} * \vec{c} \in \mathcal{G}(L)$ it follows that L/K is Galois.

For $\sigma \in \mathrm{Gal}(L/K)$ let $\vec{c}_\sigma$ be the unique element of $\mathcal{G}(\mathbb{F}_p)$ such that $\sigma(\vec{x}) = \vec{x} * \vec{c}_\sigma$. The map $\sigma \mapsto \vec{c}_\sigma$ gives a one-to-one homomorphism from $\mathrm{Gal}(L/K)$ into $\mathcal{G}(\mathbb{F}_p)$.

Therefore $\vec{x}$ determines a homomorphism $\theta_{\vec{x}} : \mathrm{Gal}(K^{sep}/K) \to \mathcal{G}(\mathbb{F}_p)$, with $\theta_{\vec{x}}(\sigma) = \vec{c}_\sigma$.

Classifying Galois representations

Let $\vec{x}$ and $\vec{y}$ be solutions to $\phi(\vec{X}) = \vec{\beta} * \vec{X}$. Then there is $\vec{d} \in \mathcal{G}(\mathbb{F}_p)$ such that $\vec{y} = \vec{x} * \vec{d}$. Hence for $\sigma \in \mathrm{Gal}(K^{sep}/K)$ we have

$$\vec{y} * \theta_{\vec{y}}(\sigma) = \sigma(\vec{y}) = \sigma(\vec{x} * \vec{d}) = \vec{x} * \theta_{\vec{x}}(\sigma) * \vec{d} = \vec{y} * \vec{d}^{-1} * \theta_{\vec{x}}(\sigma) * \vec{d}.$$

Therefore $\theta_{\vec{y}}(\sigma) = \vec{d}^{-1} * \theta_{\vec{x}}(\sigma) * \vec{d}$. It follows that $\vec{\beta}$ determines a conjugacy class of homomorphisms from $\mathrm{Gal}(K^{sep}/K)$ to $\mathcal{G}(\mathbb{F}_p)$.

Let $\vec{\alpha} \in \mathcal{G}(K)$ and set $\vec{\beta}' = \phi(\vec{\alpha}) * \vec{\beta} * \vec{\alpha}^{-1}$. If $\vec{x}$ is a solution to $\phi(\vec{X}) = \vec{\beta} * \vec{X}$ then $\vec{z} := \vec{\alpha} * \vec{x}$ is a solution to $\phi(\vec{X}) = \vec{\beta}' * \vec{X}$. It follows that $\theta_{\vec{z}} = \theta_{\vec{x}}$.

Define an equivalence relation on $\mathcal{G}(K)$ by $\vec{\beta} \sim \vec{\beta}'$ if there is $\vec{\alpha} \in \mathcal{G}(K)$ such that $\vec{\beta}' = \phi(\vec{\alpha}) * \vec{\beta} * \vec{\alpha}^{-1}$. Then we have

Theorem (Galois classification theorem)

Let $\mathcal{G}$ be a filtered unipotent group over $\mathbb{F}_p$. Then there is a one-to-one correspondence between equivalence classes $[\vec{\beta}]$ of elements of $\mathcal{G}(K)$ and conjugacy classes of homomorphisms from Gal(K^{sep}/K) to $\mathcal{G}(\mathbb{F}_p)$, which maps $[\vec{\beta}]$ to the conjugacy class of $\theta_{\vec{x}}$ for any $\vec{x}$ such that $\phi(\vec{x}) = \vec{\beta} * \vec{x}$.

Witt vectors

Let W_n denote the p-Witt vectors of length n. Then the Witt vector addition operation $\oplus$ makes W_n an n-dimensional filtered unipotent group over $\mathbb{F}_p$.

Let $\vec{\beta}, \vec{\beta}' \in W_n(K)$. Then $\vec{\beta} \sim \vec{\beta}'$ if and only if there is $\vec{\alpha} \in W_n(K)$ such that $\vec{\beta}' = \phi(\vec{\alpha}) \oplus \vec{\beta} \ominus \vec{\alpha}$.

Thus there is a one-to-one correspondence between equivalence classes $[\vec{\beta}]$ with $\vec{\beta} \in W_n(K)$ and homomorphisms

$$\theta : \mathrm{Gal}(K^{sep}/K) \longrightarrow W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

In particular, by taking $n = 1$ we recover Artin-Schreier theory.

The Heisenberg group

Let $p > 2$. The Heisenberg group G is isomorphic to $\mathcal{G}(\mathbb{F}_p)$, where $\mathcal{G}$ is the algebraic group over $\mathbb{F}_p$ whose K^{sep}-points are

$$\mathcal{G}(K^{sep}) = \left\{ \begin{bmatrix} 1 & c_1 & c_3 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix} : c_i \in K^{sep} \right\},$$

with the operation of matrix multiplication. Then $\mathcal{G}$ is a 3-dimensional filtered unipotent group over $\mathbb{F}_p$.

The Heisenberg group ...

Let $\vec{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathcal{G}(K)$ and let $\vec{x}$ satisfy $\phi(\vec{x}) = \vec{\beta} * \vec{x}$. Since

$$\begin{bmatrix} 1 & \beta_1 & \beta_3 \\ 0 & 1 & \beta_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_1 + x_1 & \beta_3 + x_3 + \beta_1 x_2 \\ 0 & 1 & \beta_2 + x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

we get

$$\vec{\beta} * \vec{x} = (\beta_1 + x_1, \beta_2 + x_2, \beta_3 + x_3 + \beta_1 x_2).$$

It follows that $\phi(\vec{x}) = \vec{\beta} * \vec{x}$ if and only if the entries of $\vec{x} = (x_1, x_2, x_3)$ satisfy

$$\begin{aligned} x_1^p - x_1 &= \beta_1 \\ x_2^p - x_2 &= \beta_2 \\ x_3^p - x_3 &= \beta_3 + \beta_1 x_2. \end{aligned}$$

Filtered pro-unipotent groups

A *filtered pro-unipotent group* $\mathcal{G}$ over $\mathbb{F}_p$ is given by a sequence of polynomials

$$D_i \in \mathbb{F}_p[X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_{i-1}]$$

such that for each $n \geq 1$

$$\vec{X} *_n \vec{Y} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} *_n \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} X_1 + Y_1 \\ X_2 + Y_2 + D_2 \\ \vdots \\ X_n + Y_n + D_n \end{bmatrix}$$

defines a filtered unipotent group over $\mathbb{F}_p$.

The operations $*_n$ for $n \geq 1$ combine to give a group operation on $\mathcal{G}$ which we denote by $*$.

Filtered pro-unipotent groups and Galois representations

Let $(\mathcal{G}, *)$ be a filtered pro-unipotent group and let $\vec{\beta} = (\beta_1, \beta_2, \dots) \in \mathcal{G}(K)$. Then there exists $\vec{x} = (x_1, x_2, \dots) \in \mathcal{G}(K^{sep})$ such that $\phi(\vec{x}) = \vec{\beta} * \vec{x}$. Set $L = K(x_1, x_2, \dots)$; then L is independent of the choice of $\vec{x}$.

For $\sigma \in \mathrm{Gal}(L/K)$ there is unique $\vec{c}_\sigma \in \mathcal{G}(\mathbb{F}_p)$ such that $\sigma(\vec{x}) = \vec{x} * \vec{c}_\sigma$. The map $\theta_{\vec{x}} : \mathrm{Gal}(K^{sep}/K) \to \mathcal{G}(\mathbb{F}_p)$ defined by $\theta_{\vec{x}}(\sigma) = \vec{c}_\sigma$ induces a one-to-one homomorphism from $\mathrm{Gal}(L/K)$ into $\mathcal{G}(\mathbb{F}_p)$.

As in the finite-dimensional setting, $\vec{\beta}$ determines a conjugacy class of homomorphisms from $\mathrm{Gal}(K^{sep}/K)$ to $\mathcal{G}(\mathbb{F}_p)$.

Define an equivalence relation on $\mathcal{G}(K)$ by $\vec{\beta} \sim \vec{\beta}'$ if there is $\vec{\alpha} \in \mathcal{G}(K)$ such that $\vec{\beta}' = \phi(\vec{\alpha}) * \vec{\beta} * \vec{\alpha}^{-1}$.

The Galois classification theorem applies here: There is a one-to-one correspondence between equivalence classes $[\vec{\beta}]$ of elements of $\mathcal{G}(K)$ and conjugacy classes of homomorphisms from $\mathrm{Gal}(K^{sep}/K)$ to $\mathcal{G}(\mathbb{F}_p)$ which maps $[\vec{\beta}]$ to the conjugacy class of $\theta_{\vec{x}}$ for any $\vec{x}$ such that $\phi(\vec{x}) = \vec{\beta} * \vec{x}$.

Witt vectors again

Let W denote the full ring of p-Witt vectors. Then $(W, \oplus)$ is a filtered pro-unipotent group over $\mathbb{F}_p$.

Let $\vec{\beta}, \vec{\beta}' \in W(K)$. Then $\vec{\beta} \sim \vec{\beta}'$ if and only if there is $\vec{\alpha} \in W(K)$ such that $\vec{\beta}' = \phi(\vec{\alpha}) \oplus \vec{\beta} \ominus \vec{\alpha}$.

Thus there is a one-to-one correspondence between equivalence classes $[\vec{\beta}]$, with $\vec{\beta} \in W(K)$, and homomorphisms

$$\theta : \mathrm{Gal}(K^{sep}/K) \longrightarrow W(\mathbb{F}_p) \cong \mathbb{Z}_p.$$

Lie algebras and p-groups

Let $\mathcal{L}$ be a Lie algebra over $\mathbb{F}_p$ which is nilpotent of class $c < p$.

The Baker-Campbell-Hausdorff formula defines a group operation on $\mathcal{L}$. This operation is expressed in terms of the Lie algebra operations $+$ and $[\ ,\ ]$:

$$x * y = x + y + \frac{1}{2} \cdot [x, y] + \frac{1}{12} \cdot ([x, [x, y]] - [y, [x, y]]) + \cdots .$$

Since $\mathcal{L}$ is nilpotent of class $c < p$, the Baker-Campbell-Hausdorff formula for $\mathcal{L}$ has only finitely many terms. The coefficients are rational numbers whose denominators are not divisible by p.

The operation $*$ makes $\mathcal{L}$ a group with exponent p and nilpotence class c.

Let $1 \leq d < p$. This construction defines an equivalence between the category of Lie algebras over $\mathbb{F}_p$ with nilpotence class $\leq d$ and the category of groups G with nilpotence class $\leq d$ such that $g^p = 1$ for all $g \in G$.

Lie algebras and filtered (pro-)unipotent groups

Let $\mathcal{L}$ be a finite Lie algebra over $\mathbb{F}_p$ with nilpotence class $c < p$. Then $\mathcal{L} \otimes_{\mathbb{F}_p} K^{sep}$ is a Lie algebra over K^{sep}, also with nilpotence class c.

Let $*$ be the operation on $\mathcal{L} \otimes_{\mathbb{F}_p} K^{sep}$ defined by the Baker-Campbell-Hausdorff formula and set $\mathcal{G} = (\mathcal{L} \otimes_{\mathbb{F}_p} K^{sep}, *)$. Then $\mathcal{G}(\mathbb{F}_p) \cong (\mathcal{L}, *)$.

Since $* : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is is a polynomial map, by choosing an appropriate $\mathbb{F}_p$-basis for $\mathcal{L}$ we can make $\mathcal{G}$ a filtered unipotent group.

Now suppose that $\mathcal{L}$ is an inverse limit of a sequence $\mathcal{L}_1, \mathcal{L}_2, \dots$ of Lie algebras over $\mathbb{F}_p$ such that $\dim_{\mathbb{F}_p}(\mathcal{L}_n) = n$. In this case, the choice of an appropriate topological $\mathbb{F}_p$-basis for $\mathcal{L}$ makes $\mathcal{G}$ a filtered unipotent group, with $\mathcal{G}(\mathbb{F}_p) \cong (\mathcal{L}, *)$.

In either case we can use the Galois classification theorem to determine the conjugacy classes of Galois representations

$$\theta : \mathrm{Gal}(K^{sep}/K) \longrightarrow \mathcal{G}(\mathbb{F}_p).$$

in terms of equivalence classes of elements of $\mathcal{G}(K)$.

References

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